The nonlinear sigma model and spin ladders

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Abstract. The well known Haldane map from spin chains into the $O(3)$ nonlinear sigma model is generalized to the case of spin ladders. This map allows us to explain the different qualitative behaviour between even and odd ladders, in exactly the same way as it explains the difference between integer and half-integer spin chains. Namely, for even ladders the topological term in the sigma model action is absent, while for odd ladders the θ parameter, which multiplies the topological term, is equal to $2\pi S$, where S is the spin of the ladder. Hence even ladders should have a dynamically generated spin gap, while odd ladders with half-integer spin should stay gapless and physically equivalent to a perturbed $SU(2)_1$ Wess–Zumino–Witten model in the infrared regime. We also derive some consequences from the dependence of the sigma model coupling constant on the ladder Heisenberg couplings constants.

1. Introduction

One of the most studied field theories in two dimensions is the nonlinear sigma model. From a particle physics point of view these models are ideal analogues of four-dimensional quantum chromodynamics, since they display asymptotic freedom behaviour [1], dynamical mass generation, and the existence of instantons [2]. In string theory the conformal invariant sigma models are crucial for understanding the on-shell properties on the strings [3]. In solid state physics the $O(3)$ sigma model also plays an important role in understanding the properties of spin systems in various dimensions. The map from spin chains into the sigma model led Haldane to the celebrated gap of antiferromagnetic Heisenberg chains with integer values of spin [4, 5]. The crucial parameter which controls the behaviour of the sigma model is the angle θ that multiplies a topological term into the action and which, according to the map, takes the value $\theta = 2\pi S$, where S is the spin of the chain. Haldane's prediction followed from the fact that the sigma models with $\theta = 0 (\mathrm{mod} 2\pi)$ are massive field theories [6–8]. This map was deduced in the semiclassical limit where $S \gg 1$, but there now is clear experimental and theoretical evidence of the existence of the gap [9, 10]. For half-integer values of the spin, the prediction, based on the gapless character of the spin-$\frac{1}{2}$ chain [11], was that all these models should also be gapless. This has been confirmed by numerical computations [12]. The sigma model at $\theta = \pi$ has also been proved to be massless [13].

Nowadays there is a better understanding of the sigma model at $\theta = \pi$ by means of the powerful techniques of bosonization [14, 15] and conformal field theory, and also from the factorized scattering theory [16, 17]. It has been shown that the low-energy physics of the $\theta = \pi$ model is well described by the $SU(2)_1$ Wess–Zumino–Witten (WZW) model

[18, 19]. The asymptotically free theory in the ultraviolet region, which is described by two goldstone modes, becomes a $SU(2)_1$ WZW model in the infrared region [16]. This RG flow satisfies the Zamolodchikov c-theorem [20]. An alternative description of the sigma model for low energies is given by a marginal irrelevant perturbation of the WZW model by the product of the two chiral $SU(2)$ currents [18]. In this manner one can perform perturbative calculations around the conformal point [21]. It is interesting to observe that there are no relevant perturbations of the $SU(2)_1$ WZW model, which explains why this conformal field theory characterizes the universality class of a large variety of spin systems.

We shall show in this paper that the sigma model methods can be extended to spin ladders, which will then allow us to consider certain questions arising in this subject. Spin ladders are arrangements of n_ℓ parallel spin chains with nearest-neighbour Heisenberg couplings between the spins along and across the chains. These spin systems are interpolating structures between one and two dimensions. The interest in spin ladders increased enormously when experimentalists discovered materials such as $(VO)_2P_2O_7$ [23] and $Sr_{n-1}Cu_{n+1}O_{2n}$ [24], whose magnetic and electronic structure was analysed in [25]. Hence, from an experimental and theoretical point of view the spin ladders have become a place to test different ideas concerning strongly correlated systems ([26] for a review on the subject).

A central question in the study of spin ladders is their different qualitative behaviour as a function of the number of legs n_ℓ. The main conjecture is that ladders with an even number of legs have a finite spin gap and short distance correlations, while the odd ladders have gapless spin excitations and power-law correlations. Many authors have contributed to clarify this question and, despite some initial controversies, it is now clear that it must be correct [27–35]. In this paper we shall give further support of this conjecture using sigma model techniques, which will clearly show the topological nature of the mechanism underlying the existence or absence of a spin gap in the ladder's spectrum. Our proof is an extension of Haldane's result from the chain to the ladder. Indeed, we shall show that the low-lying modes of the ladder are described by a sigma model with a value of the angle θ equal to zero for the even chains and equal to $2\pi S$ for the odd ones. Hence, from this result and the well known properties of the sigma model at $\theta = 0$ and π, we prove the above conjecture. This kind of proof has been suggested in [34] on the basis of the two-dimensional (2D) formulation of the sigma model due to Haldane, who showed the absence of the topological term in the 2D Heisenberg antiferromagnet [36]. However, our approach to the problem is not really 2D since we take into account the specific nature of the ladders (i.e. objects in between 1D and 2D). This different treatment is made clear by the nature of the sigma model variables that we use which are 1D fields.

In the case of odd ladders with half-integer spins there is an extension of the Lieb–Mattis–Schultz theorem due to Affleck (the LMSA thorem) which states that in the infinite length limit either the ground state is degenerate or else there are gapless excitations [37, 38]. Our results, together with those of [25, 29, 30, 32, 33] confirm that the last possibility is the one realized by the spin ladders, the spin chain being the particular case $n_\ell = 1$. The LMSA theorem works under very general circumstances, which is a manifestation of the topological nature of odd ladders and half-integer spins. To make this more transparent we shall consider spin ladders with arbitrary values of the interchain and intrachain coupling constants. Our results concerning the nature of the gap will be independent of the precise values taken by these parameters. We want to mention here another topological interpretation of the difference between even and odd ladders given in [29] in the framework of the RVB picture [39]. According to [29] the even ladders are short-range RVB systems which have a gap due to the confinement of topological defects, while the odd ladders are long-range RVB

systems with no confinement and consequently no gap. It would be interesting to analyse the relation between these two topological interpretations.

2. AF spin ladders: goldstone modes

The Hamiltonian of a spin ladder with n_ℓ legs of length N is given by

$$H_{\text{ladder}} = H_{\text{leg}} + H_{\text{rung}} \tag{1}$$

$$H_{\text{leg}} = \sum_{a=1}^{n_\ell} \sum_{n=1}^{N} J_a S_a(n) \cdot S_a(n+1)$$

$$H_{\text{rung}} = \sum_{a=1}^{n_\ell-1} \sum_{n=1}^{N} J'_{a,a+1} S_a(n) \cdot S_{a+1}(n)$$

where $S_a(n)$ are spin-S matrices located in the ath leg at the position $n = 1, \ldots, N$. We consider periodic boundary conditions along the legs ($S_a(n) = S_a(n+N)$). The only condition we shall impose on the coupling constants J_a and $J'_{a,a+1}$ is that they are positive, which guarantees that the H_{ladder} possesses, in the classical limit ($S \to \infty$), a minima given by the antiferromagnetic vacuum solution,

$$S_a^{\text{class}}(n) = (-1)^{a+n} S z \tag{2}$$

where z is the unit vector in the vertical direction. Solution (2) breaks the $O(3)$ rotational invariance of the H_{ladder} down to the subgroup $O(2)$ of rotations around the z-axis. Consequently there should appear two goldstone modes associated with the broken generators S^x and S^y [5]. In the thermodynamic limit where $N \to \infty$, with n_ℓ kept fixed, the spectrum of the Hamiltonian (1) becomes essentially 1D, despite its 2D origin, and hence one expects that the quantum corrections will restore the $O(3)$ symmetry, as happens for the usual Heisenberg chain. Our strategy will then parallel the one used for the study of spin chains. We shall only consider the massless degrees of freedom associated with the two goldstone modes and later on we shall consider their interaction in the framework of the sigma model. A direct way to find the goldstone modes, which are nothing but spin waves, is through the linearized approximation of the equation of motion of the spins [40].

The evolution equation of the spin operators of the ladder are given by

$$\frac{\mathrm{d}S_a(n)}{\mathrm{d}t} = \mathrm{i}\,[H_{\text{ladder}}, S_a(n)] = -S_a(n)$$
$$\times \left[J_a\left(S_a(n+1) + S_a(n-1)\right) + J'_{a,a+1} S_{a+1}(n) + J'_{a,a-1} S_{a-1}(n) \right]. \tag{3}$$

This equation is valid for any $a = 1, \ldots, n_\ell$ with the convenium $J'_{0,1} = J'_{n_\ell,n_\ell+1} = 0$ and $J'_{a,b} = J'_{b,a}$. Expanding $S_a(n)$ around the classical solution (2),

$$S_a(n) = S_a^{\text{class}}(n) + s_a(n) \tag{4}$$

one gets, in the linearized approximation,

$$\frac{\mathrm{d}\zeta_a(n)}{\mathrm{d}t} = \mathrm{i}(-1)^{a+n+1} S\left[J_a(\zeta_a(n+1) + \zeta_a(n-1) + 2\zeta_a(n)) + \sum_b K_{a,b}^+ \zeta_b(n) \right] \tag{5}$$

where $\zeta_a(n) = s_a^x(n) + \mathrm{i}s_a^y(n)$ and the matrix $K_{a,b}^+$ together with a matrix $K_{a,b}^-$, which we shall use later on, are defined as follows:

$$K_{a,b}^{\pm} = \begin{cases} J'_{a,a+1} + J'_{a,a-1} & a = b \\ \pm J'_{a,b} & |a-b| = 1. \end{cases} \tag{6}$$

The solution of equation (5) is given by plane waves,

$$\zeta_a(n) = e^{i(\omega t + kn)}(\psi_a(k) + (-1)^{a+n+1}\phi_a(k)). \tag{7}$$

Introducing (7) into (5) one gets

$$\frac{\omega}{S}\psi_a(k) = 4\sin^2\frac{k}{2}J_a\phi_a(k) + \sum_b K^-_{a,b}\phi_b(k) \tag{8}$$

$$\frac{\omega}{S}\phi_a(k) = 4\cos^2\frac{k}{2}J_a\psi_a(k) + \sum_b K^+_{a,b}\psi_b(k).$$

These equations have massless and massive modes in the limit $N \to \infty$ (periodicity along the legs implies $k = 2\pi m/N$, $m = 0, 1, \ldots, N-1$). These modes can be obtained by expanding $\omega(k)$, $\psi_a(k)$ and $\phi_a(k)$ in powers of the momenta k. For the massless modes this expansion reads

$$\omega = vk + O(k^3)$$
$$\psi_a(k) = kA_a + O(k^3) \tag{9}$$
$$\phi_a(k) = B_a + k^2 C_a + O(k^4).$$

The equations for the coefficients v, A_a, B_a and C_a follow from (8),

$$0 = \sum_b K^-_{a,b}B_b \tag{10}$$

$$\frac{v}{S}B_a = \sum_b L_{a,b}A_b \tag{11}$$

$$\frac{v}{S}A_a = J_a B_a + \sum_b K^-_{a,b}C_b \tag{12}$$

where we have introduced yet another matrix $L_{a,b}$ given by

$$L_{a,b} = 4J_a\delta_{a,b} + K^+_{a,b} \tag{13}$$

which is going to play an important role in the construction. Notice that $L_{a,b}$ is a positive definite matrix.

The solution of equation (10) is uniquely given by $B_a = B \; \forall a$. This result follows from the observation that $K^-_{a,b}$ is a generalized incidence matrix associated with a graph consisting of n_ℓ points labelled by a and links joining the points a and b whenever $J'_{a,b}$ is non-null. Since the graph is connected (i.e. $J'_{a,b}$ is a non-singular matrix) there is a unique vector satisfying equation (10). Connectedness of the graph simply means that the ladder cannot be split into two or more subladders. This graph, together with its incidence matrix, contains all the information concerning the rungs of the spin ladder relevant to our problem.

The solution of (11) is given by

$$A_a = \frac{v}{S}B\sum_b L^{-1}_{a,b} \tag{14}$$

where we have inverted the matrix L (recall that L is positive definite). To solve equation (12) we proceed in two steps. First of all we sum over the index a in (12) and use the fact that $\sum_a K^-_{a,b} = 0$, to get rid of the term proportional to C_a. This gives us an equation for the spin wave velocity v, which, with the help of (14), can be written as,

$$\left(\frac{v}{S}\right)^2 = \frac{\sum_a J_a}{\sum_{a,b} L^{-1}_{a,b}}. \tag{15}$$

Both the numerator and the denominator of (15) are positive which yields a real value for v/S.

The solution of (12) for the vector C_a constitutes in fact a one-parameter family of solutions given by $C_a + x B_a$ with x arbitrary. This freedom reflects the linearity of equations (8). Multiplying the whole solution (7) by a k-dependent factor produces a term of the form $k^2 B_a$ in (9). The 'transverse' components of C_a can then be obtained by inverting the K^- matrix in the subspace orthogonal to its zero eigenvector.

Next we shall briefly consider the massive modes. The value of the gap $\Delta = \omega(k = 0)$ can be simply obtained by setting $k = 0$ in (8),

$$\frac{\Delta}{S}\psi_a(0) = \sum_b K_{a,b}^- \phi_b(0) \tag{16}$$

$$\frac{\Delta}{S}\phi_a(0) = \sum_b L_{a,b}\psi_b(0).$$

Hence combining both equations we find that $(\Delta/S)^2$ is given by the eigenvalues of the matrix LK^- or alternatively K^-L. One of these eigenvalues is zero and corresponds to the massless mode studied above, and the others are all non-zero and positive corresponding to the massive modes.

3. σ-model mapping

Let us recall how one maps the Heisenberg spin chain into the 1D σ-model (we shall follow [5] closely). The spin wave analysis shows that the spin operators $S(n)$ have two smooth centred components and momenta $k = 0$ and $k = \pi$, which can be identified with the total angular momenta l and the staggered field φ, respectively. The relation between these operators can be written as follows,

$$S(2n) = l(x) - S\varphi(x) \tag{17}$$

$$S(2n + 1) = l(x) + S\varphi(x)$$

where $x = 2n + \frac{1}{2}$ is the midpoint coordinate of the block formed by the points $2n$ and $2n + 1$. Inverting equation (17) one gets

$$l(x) = (S(2n + 1) + S(2n))/2 \tag{18}$$

$$\varphi(x) = (S(2n + 1) - S(2n))/2S.$$

l and φ satisfy the following commutation relations,

$$[l^i(x), l^j(y)] = i\frac{\delta_{x,y}}{2}\epsilon^{ijk}l^k(x)$$

$$[l^i(x), \varphi^j(y)] = i\frac{\delta_{x,y}}{2}\epsilon^{ijk}\varphi^k(x) \tag{19}$$

$$[\varphi^i(x), \varphi^j(y)] = i\delta_{x,y}\epsilon^{ijk}l^k(x)/2S^2 \to 0$$

which can be derived from the $SU(2)$ algebra satisfied by the spin operators $S(n)$. The term $\delta_{x,y}/2$ is the lattice version of Dirac's delta function $\delta(x - y)$. The factor two in the denominator is simply the lattice spacing of the 2-block arrangement of the chain. The fact that $S(n)$ are spin-S matrices satisfying the relation $S^2(n) = S(S + 1)$, implies two additional equations for l and φ, namely

$$\varphi^2 = 1 - l^2/S^2 + O(1/S) \to 1 \tag{20}$$

$$l \cdot \varphi = 0.$$

Equations (19) and (20) are the standard relations satisfied by the σ-field φ and the angular momenta l. Introducing equations (17) into the spin chain Heisenberg Hamiltonian one gets, taking the continuum limit and eliminating higher derivatives terms, the standard σ-model Hamiltonian,

$$H_\sigma = \frac{v_\sigma}{2} \int dx \left[g \left(l^2 - \frac{\theta}{4\pi} \varphi' \right)^2 + \frac{1}{g} \varphi'^2 \right] \tag{21}$$

where $\varphi' = \partial_x \varphi$. The theta parameter, coupling constant and spin velocity take the following values:

$$\theta = 2\pi S \qquad g = \frac{2}{S} \qquad v_\sigma = 2JS \qquad (n_\ell = 1). \tag{22}$$

The Hamiltonian (21) can be obtained from the 2D σ-model Lagrangian:

$$L = \frac{1}{2g} \partial_\mu \varphi \cdot \partial_\mu \varphi + \frac{\theta}{8\pi} \epsilon^{\mu\nu} \varphi \cdot (\partial_\mu \varphi \times \partial_\nu \varphi). \tag{23}$$

It is our aim to generalize the previous construction to spin ladders. First of all we shall divide the ladder into blocks of two consecutive rungs and define σ-model variables for each of them. The spin wave analysis of the previous section suggests the following ansatz,

$$S_a(2n) = A_a l + l_a + S(-1)^a (\varphi + \varphi_a) \tag{24}$$
$$S_a(2n+1) = A_a l + l_a + S(-1)^{a+1} (\varphi + \varphi_a)$$

where l and φ are the candidates for the σ variables, and l_a and φ_a are some extra slowly-varying fields needed to match the number of degrees of freedom on both sides of (24). For this to be the case we shall impose the following 'transversality conditions' on l_a and φ_a,

$$\sum_a l_a = 0 \qquad \sum_a \varphi_a = 0 \tag{25}$$

which are only needed for $n_\ell > 1$. Using (25) we can express l and φ in terms of the spin operators as follows:

$$l(x) = \sum_a [S_a(2n+1) + S_a(2n)] \bigg/ \left(2 \sum_b A_b \right) \tag{26}$$

$$\varphi(x) = \sum_a (-1)^{a+1} [S_a(2n+1) - S_a(2n)] / (2Sn_\ell).$$

We want l and φ to satisfy the algebraic relations (19), which can be achieved by imposing

$$\sum_a A_a = 1 \Rightarrow A_a = \frac{\sum_b L_{a,b}^{-1}}{\sum_{c,d} L_{c,d}^{-1}} \tag{27}$$

where we have used equation (14). Similarly l and φ as given by (26) satisfy equations similar to (20),

$$\varphi^2 = 1 + O(1/Sn_\ell) \qquad l \cdot \varphi = O(1/Sn_\ell). \tag{28}$$

Hence in the limit $Sn_\ell \gg 1$ we obtain the constraints which define the σ-model. From (28) it seems that the expansion parameter that we are employing is Sn_ℓ rather than S. If this is correct then considering higher spins is equivalent, from the sigma model point of view, to considering ladders with many legs. We shall return later on to this suggestion. Another point which is worth mentioning is that $l(x)$ and $\varphi(x)$ represent total angular momenta and staggered magnetization of the rung taken as a whole. This is why they are 1D densities

depending only on the single coordinate x. Given the relations (24) we can now write the spin ladder Hamiltonian (1) in the variables l, φ, l_a and φ_a as follows:

$$H_{\text{ladder}} = \int \frac{\mathrm{d}x}{2} \left\{ \sum_{a,b} L_{a,b}(A_a A_b l^2 + l_a l_b) + 2S^2 \sum_a J_a(\varphi' + \varphi'_a)^2 + \sum_{a,b} K^-_{a,b} \varphi_a \varphi_b \right.$$

$$\left. + 2S \sum_a (-1)^a J_a[(A_a l + l_a)(\varphi' + \varphi'_a) + (\varphi' + \varphi'_a)(A_a l + l_a)] \right\}. \tag{29}$$

To derive (29) we have used (11), (25) and the following formula,

$$(A_a l + l_a)^2 + S^2(\varphi + \varphi_a)^2 = S(S+1) \tag{30}$$

$$(A_a l + l_a)(\varphi + \varphi_a) = 0$$

which is a consequence of the relations $S_a^2(n) = S(S+1)$ (since we are working in the semiclassical limit $S \gg 1$ we shall only keep the highest power in S). To decouple the fields φ and φ_a in (29) we have to choose the same value of J_a for all the legs. Indeed, upon this condition the cross term $\sum_a J_a \varphi' \varphi'_a$ in (29) vanishes as a consequence of (25). We thus obtain that φ is a massless field while the fields φ_a are massive. Let us now concentrate on the massless field, whose Hamiltonian reads

$$H_{\text{ladder}}^{(\text{massless})} = \int \frac{\mathrm{d}x}{2} \left[\left(\sum_{a,b} L_{a,b} A_a A_b \right) l^2 + 2S^2 \sum_a J_a \varphi^2 + 2S \sum_a (-1)^a J_a A_a (l\varphi' + \varphi' l) \right]. \tag{31}$$

This is precisely the σ-model Hamiltonian given in (21) with an appropiate identification of θ, g and v. Let us first consider θ whose values are given by

$$\theta = 8\pi S \frac{\sum_a (-1)^{a+1} J_a A_a}{\sum_{b,c} L_{b,c} A_b A_c}. \tag{32}$$

A simplification of (32) is achieved using (27):

$$\theta = 8\pi S \sum_{a,b} (-1)^{a+1} J_a L_{a,b}^{-1}. \tag{33}$$

A convenient way of writing (33) is by means of two n_ℓ-dimensional vectors $|F\rangle$ and $|AF\rangle$ defined as follows,

$$|F> = \frac{1}{\sqrt{n_\ell}} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \qquad |AF< = \frac{1}{\sqrt{n_\ell}} \begin{pmatrix} 1 \\ -1 \\ \vdots \\ (-1)^{n_\ell+1} \end{pmatrix} \tag{34}$$

whose scalar product is

$$\langle AF|F \rangle = \begin{cases} 0 & n_\ell : \text{even} \\ 1/n_\ell & n_\ell : \text{odd}. \end{cases} \tag{35}$$

Using (34) we write (33) in matrix notation as

$$\theta = 8\pi S n_\ell \langle AF | \boldsymbol{J} \boldsymbol{L}^{-1} | F \rangle \tag{36}$$

where $\boldsymbol{J}$ is a diagonal matrix whose entries are J_a. Recalling the well known operator identity

$$\frac{1}{A+B} = \frac{1}{A} - \frac{1}{A} B \frac{1}{A+B} \tag{37}$$

we transform (36) into

$$\theta = 2\pi S n_\ell \left\langle AF \left| \left(\mathbf{1} - K^+ \frac{1}{4J + K^+} \right) \right| F \right\rangle. \tag{38}$$

Then noticing that the vector $|AF\rangle$ is annihilated by K^+ and using equation (35) we arrive finally at

$$\theta = \begin{cases} 0 & n_\ell : \text{even} \\ 2\pi S & n_\ell : \text{odd.} \end{cases} \tag{39}$$

Taking into account the fact that θ is defined modulo 2π we can then write equation (39) simply as

$$\theta = 2\pi S n_\ell. \tag{40}$$

This result is valid for any value of the coupling constants J_a and $J_{a,a+1}$ as long as they are non-vanishing. This confirms the topological nature of the result in total agreement with the LMSA theorem. Moreover, the derivation of (40) suggests that the θ term for spin ladders is related to the transition amplitude from ferromagnetic to antiferromagnetic configurations along the rungs (35). A path integral derivation of (40), along the lines of [36], would probably throw some light on this interpretation.

Next we shall give the expressions of the σ-model coupling constant g and the velocity v_σ in terms of the ladder parameters,

$$g^{-1} = S \left[2 \sum_{a,b,c} J_a L_{b,c}^{-1} - \tfrac{1}{4} \delta_{n_\ell} \right]^{1/2} \tag{41}$$

$$\left(\frac{v_\sigma}{S} \right)^2 = 2 \frac{\sum_a J_a}{\sum_{b,c} L_{b,c}^{-1}} - \delta_{n_\ell} \frac{1}{\left(2 \sum_{b,c} L_{b,c}^{-1} \right)^2} \tag{42}$$

where δ_{n_ℓ} is equal to 1 (or 0) whenever n_ℓ is odd or even.

Compairing equations (42) and (15) for $n_\ell = 1$ we find that $v_\sigma = v$, but for $n_\ell > 1$ the two velocities, v_σ and v, do not coincide (for n_ℓ even, one has $v_\sigma = \sqrt{2}v$). We interpret this fact as a kind of interference effect between the legs of the ladder which makes the effective spin velocity v_σ differ from the spin wave velocity v. The most interesting case for practical applications is when $J_a = J$ and $J'_{a,a+1} = J'$ $\forall a$. We shall give below the values of g and v_σ in this situation:

$$g^{-1} = S \left[\frac{n_\ell^2}{2} f(n_\ell, J'/J) - \frac{1}{4} \delta_{n_\ell} \right]^{1/2} \tag{43}$$

$$v_\sigma = \frac{4J}{n_\ell g f(n_\ell, J'/J)}. \tag{44}$$

The function $f(n_\ell, J'/J)$ appearing in these formulae is defined as

$$f(n_\ell, J'/J) = \langle F | (1 + K^+/4J)^{-1} | F \rangle$$
$$= \frac{1}{n_\ell^2} \left[\delta_{n_\ell} + 2 \sum_{m=1,3,\dots,n_\ell - 1} \left(\sin \frac{\pi m}{2 n_\ell} \right)^{-2} \left(1 + \frac{J'}{J} \cos^2 \frac{\pi m}{2 n_\ell} \right)^{-1} \right]. \tag{45}$$

In the particular cases where $n_\ell = 2$ and 3 we obtain

$$
\text{for } n_\ell = 2 \qquad
\begin{cases}
g = \dfrac{1}{S\sqrt{2}} \left(1 + \dfrac{J'}{2J}\right)^{1/2} \\[3mm]
v_\sigma = 2\sqrt{2} S J \left(1 + \dfrac{J'}{2J}\right)^{1/2}
\end{cases}
\tag{46}
$$

$$
\text{for } n_\ell = 3 \qquad
\begin{cases}
g = \dfrac{2}{S} \left(\dfrac{1 + \frac{3}{4} J'/J}{17 + \frac{3}{4} J'/J}\right)^{1/2} \\[3mm]
v_\sigma = \dfrac{2JS}{3} \dfrac{\left[\left(1 + \frac{3}{4} J'/J\right)\left(17 + \frac{3}{4} J'/J\right)\right]^{1/2}}{1 + \frac{1}{12} J'/12J}.
\end{cases}
\tag{47}
$$

We have assumed in (47) that S is half-integer. From equation (43) we derive that $g(n_\ell, J'/J)$ is a monotonically increasing function of the ratio J'/J, which implies that the ladder is more disordered in the strong coupling regime than it is for weak coupling. In fact we get

$$
\lim_{J'/J \to \infty} g =
\begin{cases}
\sim \{J'/J\}^{1/2} \to \infty & n_\ell : \text{even} \\[2mm]
2/S & n_\ell : \text{odd, } S : \text{half integer} \\[2mm]
\sqrt{2}/S & n_\ell : \text{odd, } S : \text{integer.}
\end{cases}
\tag{48}
$$

This equation shows that the difference between even and odd ladders appears not only in the topological term but also in the behaviour of g as a function of the ratio J'/J in the strong coupling limit. For n_ℓ even, the sigma model enters into the strong coupling regime $g \gg 1$, which is dominated by the angular momentum term l^2 in the sigma model Hamiltonian (21). Let us suppose that we discretize the sigma model Hamiltonian at $\theta = 0$ as in [6, 13],

$$
H_\sigma = \frac{v_\sigma}{2} \sum_n \left[g l^2(n) - \frac{2}{g} \varphi(n) \cdot \varphi(n+1) + cte \right]
\tag{49}
$$

with $l(n)$ satisfying the standard angular momenta algebra such that l^2 has the spectrum $l(l+1), l = 0, 1, \ldots$. $l(n)$ gives the angular momenta of the nth rung of the ladder. Then in the strong coupling limit the ground state of (49) is obtained choosing the representation $l = 0$ for each n. The first excited state has $l = 1$ at one site and energy $g v_\sigma$, which is the value of the gap in the limit $g \gg 1$. In the case $n_\ell = 2$ we get from equations (46)

$$
v_\sigma g \simeq J' \qquad \text{for } J'/J \gg 1.
\tag{50}
$$

The second term in (49) produces shifts in the ground-state energy and also delocalizes the $l = 1$ excitation producing a band of states. The gap only vanishes at $g = 0$. These results agree with the ones obtained using very different techniques, namely numerical [27, 28], renormalization group [29], mean field [30], finite size [33] and bosonization [35]. However, in order to claim full agreement we have also to analyse what happens with the other massive modes that we discarded in the mapping of the ladder Hamiltonian into the sigma model Hamiltonian. If the mass which is generated dynamically by non-perturbative effects for the field φ is smaller than the gap associated with the massive modes φ_a then we expect that the map must be essentially correct, except for a finite renormalization of the coupling constant g and the spin velocity v_σ. These issues will be considered elsewhere.

For the odd ladders the asymptotic value of g is in agreement with (22), in the sense that the odd ladders with spin-$\frac{1}{2}$ can be thought of as single chains with a spin-$\frac{1}{2}$ and an

effective coupling constant J_{eff}. Indeed, for $S = \frac{1}{2}$ we get from (48) that $g = 4$ which is the same value we obtain in (22) for the single spin-$\frac{1}{2}$ chain.

Let us consider now the weak coupling limit $J'/J \ll 1$. From (43) and (45) we get

$$\lim_{J'/J \to 0} g = \begin{cases} \sqrt{2}/(Sn_\ell) & n_\ell : \text{even or } n_\ell : \text{odd}, \ S : \text{integer} \\ 2 \Big/ \left(S\sqrt{2n_\ell^2 - 1} \right) & n_\ell : \text{odd}, \ S : \text{half integer} \end{cases} \tag{51}$$

which implies that g depends essentially on the combination Sn_ℓ, as we anticipated in the discussion of equation (28). The isotropic case, $J = J'$, is in fact closer to the weak coupling values (51) than to the strong coupling ones (48). Thus for $Sn_\ell \gg 1$ the value of g will be small and we may use the formula $\exp(-2\pi/g)$ to estimate the value of the energy or mass scales of the system. This implies in particular that the mass gap for the even ladders with n_ℓ large will decrease as $\exp(-cte\, n_\ell)$. This agrees, at least qualitatively, with the numerical results which give a spin gap Δ_{spin} for the isotropic case equal to $0.504J$ for $n_\ell = 2$ and $\Delta_{\text{spin}} \sim 0.2J$ for $n_\ell = 4$. Thus, in the limit $n_\ell \to \infty$, the gap of the even ladders should vanish exponentially. As the odd ladders are already gapless for any number of legs, one reaches in the limit n_ℓ the same result for both even and odd chains. However, one must be careful in this limit since, as we mentioned above, the massive modes that we discarded in our mapping to a 1D sigma model are becoming more important as we increase the number of legs. A more careful analysis of these questions is needed.

4. Final considerations

The application of the sigma model techniques to spin ladders has allowed us not only to confirm the topological origin of the qualitatively different behaviour of even and odd ladders, but also to get some hints about the dependence of the physical quantities on the values of the coupling constants J and J'. Much work remains to be done in this direction, but we believe that the sigma model offers an unified, economic and consistent approach to spin ladders. The connection we have established in this paper allows us, in principle, to apply the knowledge accumulated in the past in the study of the sigma model to the understanding of spin ladders.

An interesting 'recent' result concerning the sigma models at $\theta = \pi$ is the proof of its exact integrability [16, 17], which is the parallel of the well known integrability of the sigma model at $\theta = 0$ [8]. The proof of integrability is performed in the framework of the factorized scattering theory. For $\theta = 0$ the S-matrix is formulated for a $O(3)$-triplet of massive particles, while for $\theta = \pi$ there are two $O(2)$-doublets of left and right moving massless particles, with three types of scatterings: left–left, right–right and left–right. Using the powerful techniques of the thermodynamic Bethe ansatz one can compute finite size effects of various observables. In this way one can prove that the RG flow for $\theta = \pi$ goes from the UV asymptotically free model with $c = 2$ to the IR massless $SU(2)_1$WZW model with $c = 1$, as we indicated in the introduction. The results one gets using these exact techniques agree at the perturbative level with the ones obtained from the perturbation of the WZW model by the marginal irrelevant operator $J_L J_R$ [19]. In this way one explains the logarithmic departure from the scale invariant results which can then be compared with experimental or numerical results [21]. These logarithmic corrections all depend on a mass scale Λ, which is generated dynamically in the sigma model and which for small values of g is given essentially by $1/a \exp(-2\pi/g)$, with a the lattice spacing. In the factorized S-matrix theory the parameter Λ appears explicitly in the expression of the energy and

momentum of the particles. An important problem is to derive the relation between Λ and the microscopic parameters of the model appearing in the Hamiltonian, namely n_ℓ, J and J' [41]. If the Hamiltonian happens to be integrable then one should be able to find an exact expression for Λ, but in general this will not be possible and so one has to use some approximation method. Numerical computations of thermodynamics quantities of the spin ladders and their comparison with the theoretical predictions may also be very useful in establishing this connection [42].

Acknowledgments

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